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POINTWISE VERSIONS OF THE MAXIMUM THEOREM WITH APPLICATIONS IN OPTIMIZATION

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ABSTRACT

We establish a sequential version of the Maximum Theorem which is suitable for solving general optimization problems by successive approximation, e.g. finite truncation of an "infinite" optimization problem. This can then be used to obtain convergence of optimal values and (partial) convergence of optimal solutions. In particular, we do this for general problems in infinite horizon optimization and semi-infinite programming.

1. Preliminaries

Let T and X be Hausdorff topological spaces with $\mathcal{K}(X)$ the set of compact, non-empty subsets of X. Consider a compact-valued, non-empty-valued correspondence F from T to X, i.e. a (set-valued) mapping $F : T \to \mathcal{K}(X)$. Recall [2] that for t_0 in T:

F is *upper semi-continuous* (u.s.c.) at t_0 if, for each open subset V of X such that $F(t_0) \subseteq V$, there exists an open neighborhood U of t_0 such that $F(t) \subseteq V$, for all $t \in U$.

F is *lower semi-continuous* (l.s.c.) at t_0 if, for each open subset V of X such that $F(t_0) \cap V \neq \emptyset$, there exists an open neighborhood U of t_0 in T such that $F(t) \cap V \neq \emptyset$, for all $t \in U$.

F is *continuous* at t_0 if it is u.s.c. and l.s.c. at t_0.

F is *closed* at t_0 if, whenever $x \in X$ is such that $x \notin F(t_0)$, there exist neighborhoods U of t_0 in T and V of x in X such that $F(t) \cap V = \emptyset$, for all $t \in U$.

We say that F is continuous (resp. u.s.c., l.s.c. or closed) on T if this is the case at each point of T.

The following are pointwise versions of Theorem 6 (p.112) and Theorem 4 (p.111) of [2] respectively.

Lemma 1.1. If F is u.s.c. at t_0, then it is closed at t_0.

Lemma 1.2. Suppose F is closed at t_0. If $t_n \to t_0$ in T, $x_n \to x_0$ in X and $x_n \in F(t_n)$, all n, then $x_0 \in F(t_0)$.

2. Pointwise Versions of the Maximum Theorem

We begin with a pointwise version of the original Maximum Theorem.

Theorem 2.1. Let T and X be Hausdorff topological spaces with $t_0 \in T$ and F as above. Suppose $f : T \times X \to \Re$ is continuous on $\{(t_0, x) : x \in F(t_0)\}$ and F is continuous at t_0. Define $m : T \to \Re$ and $M : T \to \mathcal{K}(X)$ by

$$m(t) = \max_{x \in F(t)} f(t,x), \qquad t \in T,$$

and

$$M(t) = \{x \in F(t) : f(t,x) = m(t)\}, \qquad t \in T.$$

Then m is continuous at t_0 and M is u.s.c. at t_0. If $M(t_0)$ is a singleton, then M is continuous at t_0.

Proof. The first part is proved as in [2, p.116] using appropriate pointwise versions of the results of section 8 of [2]. We leave the details to the interested reader. The second part follows from the fact that M is l.s.c. at t_0 if $M(t_0)$ is also a singleton.

There is a version of Theorem 2.1 which is more useful to us (under additional asumptions). Before we establish this, we require some additional topological concepts.

Suppose that $S = \{1,2,\ldots\}$ or $[1,\infty)$ with $T = S \cup \{\infty\}$ (one -point compactification) identified as a compact metric space via stereographic projection. If $F_t \subseteq X, \quad t \in T$, then, as in [2,5,6], define:

$\limsup_{t\in S} F_t =$ the set of x in X for which there exists a subsequence $\{F_{t_n}\}$ of $\{F_t\}$ and a corresponding sequence $\{x_{t_n}\}$ such that $x_{t_n} \in F_{t_n}$, all n, and $x_{t_n} \to x$, as $n \to \infty$.

$\liminf_{t\in S} F_t =$ the set of x in X for which there exists x_t in F_t, for t sufficiently large, such that $x_t \to x$, as $t \to \infty$.

$\lim_{t\in S} F_t = F_\infty$ if and only if $\limsup_{t\in S} F_t = \liminf_{t\in S} F_t = F_\infty$.

Suppose also that (X,d) is a metric space and $\mathcal{K}(X)$ is equipped with the Hausdorff metric [2,5,6] denoted by D. If $F_t \in \mathcal{K}(X), \quad t \in T$, then it is well-known [5, 6, 7] that if $F_t \to F_\infty$ in $\mathcal{K}(X)$ relative to D, as $t \to \infty$, then $\lim_{t\in S} F_t = F_\infty$. The converse is true if X is compact.

We are now ready to verify the following (generalized) sequence version of Theorem 2.1.

Theorem 2.2. Let T and X be as above with X compact. Suppose $\{f_t : t \in T\}$ is a family of continuous functions on X such that $f_t \to f_\infty$ uniformly on X, as $t \to \infty$. Suppose also that $\{F_t : t \in T\} \subseteq \mathcal{K}(X)$ and $F_t \to F_\infty$ relative to D in $\mathcal{K}(X)$, as $t \to \infty$, i.e. $\lim_{t\in S} F_t = F_\infty$. Define :

$$m_t = \max_{x\in F_t} f_t(x), \qquad t \in T,$$

and

$$M_t = \{x \in F_t : f_t(x) = m_t\}, \qquad t \in T.$$

Then $\lim_{t\to\infty} m_t = m_\infty$ and $\limsup_{t\in S} M_t \subseteq M_\infty$. If M_∞ is a singleton $\{x_\infty\}$, then $\lim_{t\in S} M_t = \{x_\infty\}$, i.e. $M_t \to \{x_\infty\}$ relative to D in $\mathcal{K}(X)$, as $t \to \infty$. In this case, $x_t \to x_\infty$, as $t \to \infty$, for *all* choices $x_t \in M_t, \quad t \in S$.

Proof. Define $f : T \times X \to \Re$ by

$$f(t,x) = f_t(x), \quad t \in T, x \in X,$$

and $F(t) = F_t, \quad t \in T$, so that $F : T \to \mathcal{K}(X)$ and $F(t) \to F(\infty)$ relative to D. Then F is continuous at ∞ by Theorem 1 of [2, p.126] (which is true pointwise). Moreover, f is continuous on $\{(\infty,x) : x \in X\}$, since $f_t \to f_\infty$ uniformly, f_∞ is continuous on X and

$$|f(\infty,x) - f(t,y)| \leq |f(\infty,x) - f(\infty,y)| + |f(\infty,y) - f(t,y)|$$

$$\leq |f_\infty(x) - f_\infty(y)| + |f_\infty(y) - f_t(y)|, \quad t \in S, y \in X.$$

Hence, by Theorem 2.1, m is continuous at ∞, i.e. $m_t \to m_\infty$ and M is u.s.c. at ∞. Consequently, M is also closed at ∞ (Lemma 1.1).

Now suppose $x \in \limsup_{t\in S} M_t$. Then there exists a subsequence $\{t_n\}$ of S such that $t_n \to \infty$ and a corresponding sequence $\{x_n\}$ such that $x_n \in M_{t_n}$, all n, and $x_n \to x$, as $n \to \infty$. Since M is closed at ∞, it follows that $x \in M_\infty$ (Lemma 1.2), i.e. $\limsup_{t\in S} M_t \subseteq M_\infty$.

If $M_\infty = \{x_\infty\}$, then M is continuous at ∞ by Theorem 2.1, so that $M_t \to M_\infty$ relative to D [2, p.126], as $t \to \infty$, i.e. $\lim_{t\to\infty} M_t = M_\infty$.

Finally, suppose $x_t \in M_t$, for each $t \in S$. If $\{x_t\}$ doesn't converge to x_∞, then there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ which is bounded away from x_∞. Since X is compact, passing to a subsequence if necessary, we may assume that there exists x in X such that $x_{t_n} \to x$, as $n \to \infty$. Consequently, $x \in \limsup_{t \in S} M_t$, i.e. $x = x_\infty$. Contradiction.

If we let $\mathcal{P}_t$, for $t \in T$, denote the optimization problem $\max_{x \in F_t} f_t(x)$, and view $\mathcal{P}_t$, for $t \in S$, as an approximation to $\mathcal{P}_\infty$, then Theorem 2.2 says that:

(1) The optimal objective values of the $\mathcal{P}_t$ converge to the optimal objective value of $\mathcal{P}_\infty$ (value convergence).

(2) The sets of optimal solutions of the $\mathcal{P}_t$ partially converge to the optimal solution set of $\mathcal{P}_\infty$.

(3) If $\mathcal{P}_\infty$ has a unique solution, then convergence takes place in (2) relative to the Hausdorff metric. In this case, any corresponding (generalized) sequence of optimal solutions to the $\mathcal{P}_t$ converges to an optimal solution of $\mathcal{P}_\infty$ (solution convergence).

Remarks. The reader should note the similarity between our Theorem 2.2 and Theorem 2.1 of Fiacco [4], which is formulated more generally and proved differently. Under our hypotheses, the (generalized) sequence of problems $\{\mathcal{P}_t : t \in S\}$ converges to $\mathcal{P}_\infty$ in the sense of Definition 2.4 of [4].

3. Applications in Optimization

First we consider a general infinite horizon optimization problem which includes those studied in [1, 3, 8]. In each of these cases, the problem $\mathcal{P}_\infty$ is of the form $\min_{x \in X} C(x)$, where X is the space of feasible infinite horizon strategies (a compact metric space) and $C(x)$ is the infinite horizon discounted cost of strategy x relative to some fixed underlying interest rate (a continuous real-valued function on X). We let C^* denote the optimal infinite horizon discounted cost and X^* the space of optimal infinite horizon strategies.

Now let $S = \{1, 2, \ldots\}$ or $[1, \infty)$ (discrete time or continuous time). For each $t \in S$, let $C_t(x)$ denote the t-horizon discounted cost of strategy x . In each of the above, C_t is a continuous function on X for each $t \in S$. However, it is unlikely (particularly in [8]) that $(t, x) \to C_t(x)$ is jointly continuous on $T \times X$. This is what motivated our need for pointwise versions of the Maximum Theorem. The t-horizon optimization problem $\mathcal{P}_t$ is then given by $\min_{x \in X} C_t(x)$, i.e. in each of [1, 3, 8], the t-horizon feasible region X_t is assumed to be all of X, $t \in S$. In other words, for each t, the t-horizon feasible strategies are assumed to be feasibly extendable over the infinite horizon. Moreover, in each case, under the given assumptions, it is shown that the functions C_t converge uniformly to C, as $t \to \infty$. Let C_t^* denote the optimal t-horizon discounted cost and X_t^* the subset of X consisting of t-horizon optimal solutions. Consequently, by the results of section 2, we have that:

(1) $C_t^* \to C^*$, as $t \to \infty$ [3, Lemma 4.5.4; 8, Theorem 3.2].

(2) $\limsup_{t \in S} X_t^* \subseteq X^*$ [3, Lemma 4.5.6; 8, Lemma 3.1].

(3) If $\mathcal{P}_\infty$ has a unique solution x^*, then $\lim_{t \in S} X_t^* = X^* = \{x^*\}$, i.e. $X_t^* \to X^* = \{x^*\}$ in $\mathcal{K}(X)$, as $t \to \infty$, relative to the Hausdorff metric. In this case, $x_t^* \to x^*$, as $t \to \infty$, where x_t^* is any element of X_t^*, $t \in S$ [1, Theorem 6; 3, section 4; 8, section 5].

Remark. In [1, Theorem 6], (3) above is called the Planning Horizon Theorem.

We next consider a general semi-infinite programming problem $\mathcal{P}_\infty$ of the following form:

$$\max C(x_1, \ldots, x_m)$$

subject to

$$g_k(x_1, \ldots, x_m) \leq b_k, \quad k = 1, \ldots,$$

$$u_j \leq x_j \leq v_j, \quad j = 1, \ldots, m,$$

where $C : \Re^m \to \Re$ is continuous and $g_k : \Re^m \to \Re$ is continuous, for all $k = 1, \ldots$. Let F denote the feasible region of $\mathcal{P}_\infty$ and $X = \prod_{j=1}^m [u_j, v_j]$. If we assume $F \neq \emptyset$, then $F \in \mathcal{K}(X)$. Let C^* denote the maximum value of C on F and $F^* = \{x \in F : C(x) = C^*\}$, so that $F^* \in \mathcal{K}(X)$ also. Now let $S = \{1, 2, \ldots\}$ with $T = S \cup \{\infty\}$. For each $n \in S$, let $\mathcal{P}_n$ denote the truncation of $\mathcal{P}_\infty$ given by

$$\max C(x_1, \ldots, x_m),$$

subject to

$$g_k(x_1, \ldots, x_m) \leq b_k, \quad k = 1, \ldots, n,$$

$$u_j \leq x_j \leq v_j, \quad j = 1, \ldots, m.$$

If F_n denotes the feasible region of $\mathcal{P}_n$, then $F_n \in \mathcal{K}(X)$, $F_{n+1} \subseteq F_n$, all n, and $F = \cap_{n=1}^\infty F_n$, so that $F_n \to F$ relative to the Hausdorff metric on $\mathcal{K}(X)$ [6, p.339]. As above, let C_n^* be the maximum value of C on F_n and $F_n^* = \{x \in F_n : C_n^* = C(x)\}$, $n \in S$. Then by the results of section 2, we have that:

(1) $C_n^* \to C^*$, as $n \to \infty$.

(2) $\limsup_{n \in S} F_n^* \subseteq F^*$.

(3) If $\mathcal{P}_\infty$ has a unique solution x^*, then $\lim_{n \in S} F_n^* = F^* = \{x^*\}$, i.e. $F_n^* \to F^* = \{x^*\}$, *as* $n \to \infty$, in $\mathcal{K}(X)$ relative to the Hausdorff metric. In this case, $x_n^* \to x^*$, as $n \to \infty$, where x_n^* is any element of F_n^*, $n \in S$.

Remark. In the first application, the feasible regions are constant while the cost functions vary. In the second application, the feasible regions vary while the cost functions are constant. However, our model in the previous section allows for both the cost functions and the feasible regions to vary, so long as the cost functions converge uniformly and the feasible regions converge in the Hausdorff metric.

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